Math 475

Combinatorics and Graph Theory

HW Set #1

By Bobby Rohde
2-07-2001

■ Key to Symbols and Notation

WLOG = Without Loss of Generality

${}_nP_r = P_r^n = P(n, r)$ = Number of Ordered Ways of Choosing r things from a set of n things.

${}_nC_r = C_r^n = C(n, r) = \binom{n}{r}$ = Number of Unordered Ways of Choosing r things from a set of n things.

$n! = n$ Factorial

$\mathbb{N}$ = Natural Numbers

$\mathbb{N}^0 = \mathbb{N} \bigcup \{0\}$ = Non-negative Integers

$\mathbb{Z}$ = Integers

$\mathbb{Z}^* = \mathbb{Z} - \{0\}$ = Integers Take Away 0.

$\mathbb{Q}$ = Rational Numbers

$\mathbb{Q}^* = \mathbb{Q} - \{0\}$ = Rational Numbers Take Away 0.

Problem #1

Chapter 1 - 3) Each time you move from one square to the next you must change from Black to White or vice-versa. Since there are 64 squares each traversed once, you must move from one square to the next exactly 63 times, meaning that if the first is B the last will be W, but this is impossible since the opposite corners are the same color.

Problem #2

Chapter 1 - 7) If n is odd then the cube occupies a volume of n^3 units^3 which is odd and hence can not be made of a integral number of blocks of volume 2 units^3. If n is even then $\frac{n}{2}$ is an integer and you may lay down n levels each of n rows of $\frac{n}{2}$ blocks placed end to end. Thus we may create cubes in this manner iff n is even.

Consider a block of n^3 units with n odd. Color the corners black alternating across faces with white then black in a checkboard style pattern. Choosing one outer face as top and working down, we see that the layers alternate between having one extra black and one extra white. So that the total cube has one extra black. Since each piece must cover a black and a white, we can not cover a cube minus its center unless the center is black thus evening out the number of each color. Counting clearly shows that this is true iff $n = 4k + 1$, for some integer k. Thus there exist odd numbers of the form $n = 4k + 3$ whose cubes minus center cannot be covered by such tiles.

Footnote: In particular $n = 5$ has a solution which can easily be constructed as 5 identical layers plus two pieces. Each layer is arranged as follows. Start in a corner, lay two tiles end to end such that you stretch one short of the length of the side. Turn the corner and repeat along all four edges. Now you have covered the outer edge. The inner square of 9 is covered in the same way only using 4 tiles so that the very center is left open. Stacking all 5 layers you find that there is a shaft straight down the middle. Place two tiles vertically at either end such that all but the center block in the shaft is filled in. Thus you have covered the cube of side 5 with these pieces leaving only the center open.

Problem #3

Chapter 1 - 21) Counting in from the outside, you have 3 choices for country 10, the outer ring of the square must then be colored in alternating sequence in the other 2 colors, allowing for one choice of 2 colors and country 5 at the inside is forced to be same as the outside. Thus there are 6 colorings permitted when given 3 colors.

Problem #4

Chapter 1 - 30) Piles: 22, 19, 14, 11. $22 = 10110_2$, $19 = 10011_2$, $14 = 01110_2$, $11 = 01011_2$. Thus the sum of the columns are 22242, which means the piles are balanced. Removing 6 from the 19 pile gives $13 = 01101_2$, and a sum of 13332. Thus the counter move to restore parity is to remove 14 from the pile of 22 leaving $8 = 01000_2$ and a sum of 04222.

Problem #5

Chapter 1 - 33 modified)

While both the book and the revised question seem to be driving at the same thing, they both also appear to be awkwardly worded. I am going to assume that the question is asking if the largest unbalanced bit is the j^{th} bit, show that any pile whose j^{th} bit is on in its binary repressentation can be used to balance the piles.

Let $a_1, a_2, \ldots a_t$ be the indexes of the unbalanced bits with $a_t = j$. Given any pile we may define two sets L, H such that $L = \left\{a_{i:\text{the } a_i^{\text{th}} \text{ bit in the binary representation of the pile is } 1}\right\}$ and $H = \{a_1, a_2, \ldots, a_t\} - L$. If a pile has its j^{th} bit on then $a_t \in L$. In order to balance the game we need to toggle all the bits in unbalanced columns. Thus for each $a_c \in L$ we need to subtract 2^{a_c} from the pile and for each $a_d \in H$ we must add 2^{a_d} to the pile. Since $2^j > \sum_{i=0}^{j-1} 2^i$, we know that whenever $j \in L$ then we will be subtracting more than we add, and hence the net change is a subtraction and thus an allowed move. Since each unbalanced column has experienced a change of ± 1 we now know that the sums with regard to that column have gone from odd to even and the game is balanced. QED.

Problem #6

Chapter 1 - 35) Second Player wins. Winning strategy is to always restore pile to multiple of 5. Once 95 is reached, first player must place a piece and thus make second player within winning range. The first player cannot initially reach a multiple of 5, but provided it starts at a multiple of 5, he must leave the pile sufficiently advanced that the second player can reach the next multiple of 5.

Problem #7

Chapter 2 - 8) Consider $\frac{m}{n}$ and the process of long divison. WLOG suppose that $m < n$ since the numbers ahead of the decimal point do not influence whether or not it repeats. The first decimal place, d_1, is found by expressing $10 * m = d_1 * n + r_1$, where r_1 , $d_1 \in \mathbb{N}^0$ and $r_1 < n$. Since $m < n$ $\Rightarrow$ $d_1 < 10$. From there on the succesive decimals are found by expressing $10 * r_{i-1} = d_i * n + r_i$ with the same conditions as before. However there exist only n non-negative integers less than n and thus r_i can take on only a finite number of values. Since the decimal expansion continues indefinitely, the pigeon hole principle guarantees that eventually the remainder must be the same as it was some previous time and from there on the digits must repeat. QED.

Problem #8

10 students) Consider K_{10} which each student representing a vertex and the lines colored either purple or orange, where orange denotes people who have had a class together and purple is pairs of people who have not. Thus the question is asking whether there is either a subset of K_{10} which is an orange K_3 or a purple K_4. This is identical to the question of whether $K_{10} \to K_3, K_4$ is true, however from the text we know that Ramsey number $r(3, 4) = 9 < 10$, and thus the assertion must be true.

Problem #9

Positive Integers) Clearly with only 10 choices for digits, we needn't consider positive integers with more than 10 digits. We can reasonably consider this as 10 cases, each for a seperate number of digits. In each case we insist that the foremost digit not be 0 so as to guarantee that we have a non-trivial n digit number. From there it proceeds according to multiplication principle.

- **Combinations**

1-Digit: 9 = 9
2-Digit: 9*9 = 81
3-Digit: 9*9*8 = 648
4-Digit: 9*9*8*7 = 4536
5-Digit: 9*9*8*7*6 = 27216
6-Digit: 9*9*8*7*6*5 = 136080
7-Digit: 9*9*8*7*6*5*4 = 544320
8-Digit: 9*9*8*7*6*5*4*3 = 1632960
9-Digit: 9*9*8*7*6*5*4*3*2 = 3265920
10-Digit: 9*9*8*7*6*5*4*3*2*1 = 3265920

Sum: 8877690 positive integers with distinct digits.

Problem #10

Chessboard) WLOG suppose that the upper right corner is labelled 0. Suppose a person were to proceed down the right edge and then across the bottom in such a way that each time they change squares the new number was increased by one. Then the lower left corner would have the value 14. Supposing there where no higher value on the table than 14 and none lower than 0 then define $q_0, q_1, \ldots, q_{14}$ to be the number of occurences of the numbers 0, 1, ... , 14 respectively. We know that $\sum q_i = 64$, since there are 64 squares on the board. $\frac{64}{15} > 4$, which implies by the averaging principle that $\exists$ an i such that $q_i \geq 5$. Thus we are done provided the opposing corners do represent extremes in numbering.

Suppose not, then there exists some path from the corners to a number either higher or lower. WLOG assume that there is a number less than 0 on the board. However counting towards the top from the bottom row no number could be less than 7 below the number in the 8^{th} row of a given column since each change can be at most 1. Thus the smallest number in any column can be 0. [In particular paths that curve back in a direction parrallel to one they have already travelled can not satisfy the conditions of the problem while still increasing numerically.] Hence the corners do represent the largest possible numeric extremes and we are done. QED.

Comment:

I'm typing this for two reasons, first off many grader's think my handwriting is nigh on illegible and I'm tired of being hassled. Secondly I am very proficient at the computer and probably can type this out in roughly the same amount of time it would take a normal person write it up, particularly if they are recopying their work for neatness or corrections. Of course this does make it difficult to include drawings and sketches. I may occasionally include sketches on seperate sheets at the end of what I turn in, if I feel there is an essential diagram not easily describable in words.