

CHAPTER 5

THE RADIATIVE TRANSFER EQUATION (RTE)

5.1 Derivation of RTE

Radiative transfer serves as a mechanism for exchanging energy between the atmosphere and the underlying surface and between different layers of the atmosphere. Infrared radiation emitted by the atmosphere and intercepted by satellite sensors is the basis for remote sensing of atmospheric temperature structure.

The radiance leaving the earth-atmosphere system which can be sensed by a satellite borne radiometer is the sum of radiation emissions from the earth surface and each atmospheric level that are transmitted to the top of the atmosphere. Considering the earth's surface to be a blackbody emitter (with an emissivity equal to unity), the upwelling radiance intensity I_λ for a cloudless atmosphere is given by the expression

$$I_\lambda = B_\lambda(T(p_s)) \tau_\lambda(p_s) + \sum_p \varepsilon_\lambda(\Delta p) B_\lambda(T(p)) \tau_\lambda(p)$$

where the first term is the surface contribution and the second term is the atmospheric contribution to the radiance to space. Using Kirchhoff's law, the emissivity of an infinitesimal layer of the atmosphere at pressure p is equal to the absorptance (one minus the transmittance of the layer). Consequently,

$$\varepsilon_\lambda(\Delta p) \tau_\lambda(p) = [1 - \tau_\lambda(\Delta p)] \tau_\lambda(p)$$

Since the transmittance is an exponential function of depth of the absorbing constituent,

$$\begin{aligned} \tau_\lambda(\Delta p) \tau_\lambda(p) &= \exp \left[-\sec \varphi \int_p^{p+\Delta p} k_\lambda q g^{-1} dp \right] \\ &\quad * \exp \left[-\sec \varphi \int_0^p k_\lambda q g^{-1} dp \right] \\ &= \tau_\lambda(p+\Delta p) \end{aligned}$$

Therefore

$$\varepsilon_\lambda(\Delta p) \tau_\lambda(p) = \tau_\lambda(p) - \tau_\lambda(p + \Delta p) = -\Delta \tau_\lambda(p) .$$

and

$$I_\lambda = B_\lambda(T(p_s)) \tau_\lambda(p_s) - \sum_p B_\lambda(T(p)) \Delta \tau_\lambda(p) .$$

which written in integral form reads

$$I_\lambda = B_\lambda(T(p_s)) \tau_\lambda(p_s) + \int_{p_s}^0 B_\lambda(T(p)) \frac{d\tau_\lambda(p)}{dp} dp .$$

The first term is the spectral radiance emitted by the surface and attenuated by the atmosphere, often called the boundary term and the second term is the spectral radiance emitted to space by the atmosphere.

Another approach to derivation of the RTE starts from Schwarzschild's equation written in pressure coordinates

$$dl_{\lambda} = (I_{\lambda} - B_{\lambda}) k_{\lambda} g^{-1} q \sec \varphi dp .$$

This is a first order linear differential equation, and a solution emerges when it is multiplied by an integrating factor

$$\tau_{\lambda} = \exp \left[-\sec \varphi \int_0^p g^{-1} q k_{\lambda} dp \right]$$

which has the differential

$$d\tau_{\lambda} = -\tau_{\lambda} \sec \varphi g^{-1} q k_{\lambda} dp .$$

Thus

$$\tau_{\lambda} dl_{\lambda} = -(I_{\lambda} - B_{\lambda}) d\tau_{\lambda}$$

or

$$d(\tau_{\lambda} I_{\lambda}) = B_{\lambda} d\tau_{\lambda} .$$

Integrating from p_s to 0,

$$I_{\lambda}(0) \tau_{\lambda}(0) - I_{\lambda}(p_s) \tau_{\lambda}(p_s) = \int_{p_s}^0 B_{\lambda}(T(p)) \frac{d\tau_{\lambda}(p)}{dp} dp .$$

The radiance detected by the satellite is given by $I_{\lambda}(0)$, $\tau_{\lambda}(0)$ is 1 by definition, and the surface of the earth is treated as a blackbody so $I_{\lambda}(p_s)$ is given by $B_{\lambda}(T(p_s))$. Therefore

$$I_{\lambda} = B_{\lambda}(T(p_s)) \tau_{\lambda}(p_s) + \int_{p_s}^0 B_{\lambda}(T(p)) \frac{d\tau_{\lambda}(p)}{dp} dp$$

as before. Writing this in terms of height

$$I_{\lambda} = B_{\lambda}(T(0)) \tau_{\lambda}(0) + \int_0^{\infty} B_{\lambda}(T(z)) \frac{d\tau_{\lambda}}{dz} dz .$$

$d\tau_{\lambda}/dz$ is often called the weighting function which, when multiplied by the Planck function, yields the upwelling radiance contribution from a given altitude z . An alternate form of the weighting function is $d\tau_{\lambda}/d\ln p$.

To investigate the RTE further consider the atmospheric contribution to the radiance to space of an infinitesimal layer of the atmosphere at height z ,

$$dl_{\lambda}(z) = B_{\lambda}(T(z)) d\tau_{\lambda}(z) .$$

Assume a well-mixed isothermal atmosphere where the density drops off exponentially with height

$$\rho = \rho_0 \exp(-\gamma z) ,$$

and assume k_{λ} is independent of height, so that the optical depth can be written for normal incidence

$$\sigma_{\lambda} = \int_z^{\infty} k_{\lambda} \rho \, dz = \gamma^{-1} k_{\lambda} \rho_0 \exp(-\gamma z)$$

and the derivative with respect to height

$$\frac{d\sigma_{\lambda}}{dz} = -k_{\lambda} \rho_0 \exp(-\gamma z) = -\gamma \sigma_{\lambda} .$$

Therefore, an expression for the detected radiance per unit thickness of the layer as a function of optical depth is at hand,

$$\frac{dl_{\lambda}(z)}{dz} = B_{\lambda}(T_{\text{const}}) \frac{d\tau_{\lambda}(z)}{dz} = B_{\lambda}(T_{\text{const}}) \gamma \sigma_{\lambda} \exp(-\sigma_{\lambda}) .$$

The level which is emitting the most detected radiance is given by

$$\frac{d}{dz} \left\{ \frac{dl_{\lambda}(z)}{dz} \right\} = 0 ,$$

or where $\sigma_{\lambda} = 1$. Most of the monochromatic radiance impinging upon the satellite is emitted by layers near the level of unit optical depth. Much of the radiation emanating from deeper layers is absorbed on its way up through the atmosphere, while far above the level of unit optical depth there is not enough mass to emit very much radiation. The assumption of an isothermal atmosphere with a constant absorption coefficient was helpful in simplifying the mathematics in the above derivation.

However, it turns out that for realistic vertical profiles of T and k_{λ} the above result is still at least qualitatively valid; most of the satellite detected radiation emanates from that portion of the atmosphere for which the optical depth is of order unity.

The fundamental principle of atmospheric sounding with meteorological satellites detecting the earth-atmosphere thermal infrared emission is based on the solution of the radiative transfer equation. In this equation, the upwelling radiance arises from the product of the Planck function, the spectral transmittance, and the weighting function. The Planck function consists of temperature information, while the transmittance is associated with the absorption coefficient and density profile of the relevant absorbing gases. Obviously, the observed radiance contains the temperature and gaseous profiles of the atmosphere, and therefore, the information content of the observed radiance from satellites must be physically related to the temperature field and absorbing gaseous concentration.

The mixing ratio of CO_2 is fairly uniform as a function of time and space in the atmosphere. Moreover, the detailed absorption characteristics of CO_2 in the infrared region are well-understood and its absorption parameters (i.e., half width, line strength, and line position) are known rather accurately. Consequently, the spectral transmittance and weighting functions for a given level may be calculated once the spectral interval and the instrumental response function have been given. To see the atmospheric temperature profile information, the RTE is rewritten so that

$$I_{\lambda} - B_{\lambda}(T(p_s)) \tau_{\lambda}(p_s) = \int_{p_s}^0 B_{\lambda}(T(p)) \frac{d\tau_{\lambda}(p)}{dp} \, dp .$$

It is apparent that measurements of the upwelling radiance in the CO_2 absorption band contain information regarding the temperature values in the interval from p_s to 0, once the surface temperature has been determined. However, the information content of the temperature is under the integral operator which leads to an ill-conditioned mathematical problem. This problem is

discussed further and a number of methods for the recovery of the temperature profile from a set of radiance observations in the CO₂ band are explored.

To understand the fundamental concept of remote sounding of the atmosphere, Figure 5.1 illustrates the relation between the vertical position of the spectral band weighting function and the location of the spectral band in the absorption band. The pressure broadening of the absorption band is demonstrated in the center of Figure 5.1. In the left of Figure 5.1, three separate spectral selections in the CO₂ absorption band are indicated. When the sounding radiometer views the atmosphere from above in spectral band A (at the center of the CO₂ absorption band), the radiation detected originates mainly from the higher levels of the atmosphere. Further away from the center of the absorption band, spectral band B detects radiation from lower down in the atmosphere (since the CO₂ higher in the atmosphere has a very narrow absorption line and radiation in spectral band B is transmitted). In the wing of the absorption band, little of the radiation in spectral band C emitted from the earth surface and lower atmosphere is absorbed by the CO₂ molecules higher up in the atmosphere (since the CO₂ absorption line gets narrower with height, decreasing pressure). Thus spectral band C detects radiation emitted from the lowest portion of the atmosphere (and partly from the earth surface). The contributions to the observed spectral radiances are depicted in the right part of Figure 5.1; these weighting functions move progressively lower in the atmosphere as the spectral band moves further away from the center of the absorption band. Figure 5.2 shows images of spectral measurements from the opaque to the very transmissive parts of the CO₂ absorption band.

Finally, to better understand the information regarding the gaseous concentration profile contained in the solution of the radiative transfer equation, integration by parts on the integral term yields

$$I_{\lambda} - B_{\lambda}(T(0)) = \int_{p_s}^0 \tau_{\lambda}(p) \frac{dB_{\lambda}(T(p))}{dp} dp .$$

Now, if measurements are made in the H₂O or O₃ spectral regions, and if temperature values are known, the transmittance profile may be inferred just as the temperature profile may be recovered when the spectral transmittance is given. Relating the gaseous concentration profile to the spectral transmittance, the density values are hidden in the exponent of an integral which is further complicated by the spectral integration over the response function. Because of these complications, retrieval of the gaseous density profile is very difficult. No clear-cut mathematical analyses may be followed in the solution of the density values. Therefore, the focus of subsequent sections is on the temperature inversion problem.

5.2 *Temperature Profile Inversion*

Inference of atmospheric temperature profile from satellite observations of thermal infrared emission was first suggested by King (1956). In his pioneering paper, King pointed out that the angular radiance (intensity) distribution is the Laplace transform of the Planck intensity distribution as a function of the optical depth, and illustrated the feasibility of deriving the temperature profile from the satellite intensity scan measurements.

Kaplan (1959) advanced the sounding concepts by demonstrating that vertical resolution of the temperature field could be inferred from the spectral distribution of atmospheric emission. Kaplan pointed out that observations in the wings of a spectral band sense deeper into the atmosphere, whereas observations in the band centre see only the very top layer of the atmosphere since the radiation mean free path is small. Thus, by properly selecting a set of different sounding spectral channels, the observed radiances could be used to make an interpretation of the vertical temperature distribution in the atmosphere.

Wark (1961) proposed a satellite vertical sounding programme to measure atmospheric temperature profiles. Polar orbiting sounders were first flown in 1969 and a geostationary sounder

was first launched in 1980.

In order for atmospheric temperatures to be determined by measurements of thermal emission, the source of emission must be a relatively abundant gas of known and uniform distribution. Otherwise, the uncertainty in the abundance of the gas will make ambiguous the determination of temperature from the measurements. There are two gases in the earth-atmosphere which have uniform abundance for altitudes below about 100 km, and which also show emission bands in the spectral regions that are convenient for measurement. Carbon dioxide, a minor constituent with a relative volume abundance of 0.003, has infrared vibrational-rotational bands. In addition, oxygen, a major constituent with a relative volume abundance of 0.21, also satisfies the requirement of a uniform mixing ratio and has a microwave spin-rotational band.

The outgoing radiance observed by IRIS (Infrared Interferometer and Spectrometer) on the Nimbus 4 satellite is shown in Figure 5.3 in terms of the blackbody temperature in the vicinity of the 15 μm band. The equivalent blackbody temperature generally decreases as the centre of the band is approached. This decrease is associated with the decrease of tropospheric temperature with altitude. Near about 690 cm^{-1} , the temperature shows a minimum which is related to the colder tropopause. Decreasing the wave number beyond 690 cm^{-1} , however, increases the temperature. This is due to the increase of the temperature in the stratosphere, since the observations near the band centre see only the very top layers of the atmosphere. On the basis of the sounding principle already discussed, a set of sounding wave numbers can be selected so that a temperature profile in the troposphere and lower stratosphere can be estimated. The arrows in Figure 5.3 indicate an example of such a selection.

There is no unique solution for the detailed vertical profile of temperature or an absorbing constituent because (a) the outgoing radiances arise from relatively deep layers of the atmosphere, (b) the radiances observed within various spectral channels come from overlapping layers of the atmosphere and are not vertically independent of each other, and (c) measurements of outgoing radiance possess errors. As a consequence, there are a large number of analytical approaches to the profile retrieval problem. The approaches differ both in the procedure for solving the set of spectrally independent radiative transfer equations (e.g., matrix inversion, numerical iteration) and in the type of ancillary data used to constrain the solution to insure a meteorologically meaningful result (e.g., the use of atmospheric covariance statistics as opposed to the use of an a priori estimate of the profile structure). There are some excellent papers in the literature that review the retrieval theory which has been developed over the past few decades (Fleming and Smith, 1971; Fritz *et al*, 1972; Rodgers, 1976; and Twomey, 1977). The next sections present the mathematical basis for some of the procedures which have been utilized in the operational retrieval of atmospheric profiles from satellite measurements and include some example problems that are solved by using these procedures.

5.3 Transmittance Determinations

Before proceeding to the retrieval problem, a few comments regarding the determination of transmittance are necessary.

So far, the upwelling radiance has been discussed at a monochromatic wavelength. However, for a practical instrument whose spectral channels have a finite spectral bandwidth, all quantities given in the RTE must be integrated over the bandwidth and are weighted by the spectral response of the instrument. The measured radiance over an interval λ_1 to λ_2 is given by

$$I_{\lambda\text{eff}} = \frac{\int_{\lambda_1}^{\lambda_2} \varphi(\lambda) I_{\lambda} d\lambda}{\int_{\lambda_1}^{\lambda_2} \varphi(\lambda) d\lambda}$$

where φ denotes the instrument spectral response (or slit) function and λ denotes the mean wavelength of the bandwidth. However, since B varies slowly with λ while τ varies rapidly and without correlation to B within the narrow spectral channels of the sounding spectrometer, it is

sufficient to perform the spectral integrations of B and τ independently and treat the results as if they are monochromatic values for the effective wavelength λ_{eff} .

For simplicity, we shall let the spectral response function $\phi(\lambda) = 1$ so that the spectral transmittance may be expressed by

$$\tau_{\lambda}(p) = \frac{1}{\Delta\lambda} \int_{\Delta\lambda} d\lambda \exp \left[- \frac{q}{g} \int_0^p k_{\lambda}(p') dp' \right]$$

Note that the mixing ratio q is a constant and $\Delta\lambda = \lambda_1 - \lambda_2$. In the lower atmosphere, collision broadening dominates the absorption process and the shape of the absorption lines is governed by the Lorentz profile

$$k_{\lambda} = \frac{S}{\pi} \frac{\alpha}{(\lambda - \lambda_0)^2 + \alpha^2} .$$

The half width α is primarily proportional to the pressure (and to a lesser degree to the temperature), while the line strength S also depends on the temperature. Hence, the spectral transmittance may be explicitly written as

$$\tau_{\lambda}(p) = \int_{\Delta\lambda} \frac{d\lambda}{\Delta\lambda} \exp \left[- \frac{q}{g} \int_0^p \frac{S(p')}{\pi} \frac{\alpha(p') dp'}{(\lambda - \lambda_0)^2 + \alpha^2(p')} \right] .$$

The temperature dependence of the absorption coefficient introduces some difficulties in the sounding of the temperature profile. Nevertheless, the dependence of the transmittance on the temperature may be taken into account in the temperature inversion process by building a set of transmittances for a number of standard atmospheric profiles from which a search could be made to give the best transmittances for a given temperature profile.

The computation of transmittance through an inhomogeneous atmosphere is rather involved, especially when the demands for accuracy are high in infrared sounding applications. Thus, accurate transmittance profiles are normally derived by means of line-by-line calculations, which involve the direct integration of monochromatic transmittance over the wavenumber spectral interval, weighted by an appropriate spectral response function. Since the monochromatic transmittance is a rapidly varying function of wavenumber, numerical quadrature used for the integration must be carefully devised, and the required computational effort is generally enormous.

All of the earlier satellite experiments for the sounding of atmospheric temperatures of meteorological purposes have utilized the $15 \mu\text{m}$ CO_2 band. As discussed earlier, the $15 \mu\text{m}$ CO_2 band consists of a number of individual bands which contribute significantly to the absorption. The most important of these is the ν_2 fundamental vibrational rotational band. In addition, there are several weak bands caused by the vibrational transitions between excited states, and by molecules containing less abundant isotopes.

For temperature profile retrievals, the transmittance is assumed to be determined.

5.4 Fredholm Form of RTE and the Direct Linear Inversion Method

Upon knowing the radiances from a set of sounding channels and the associated transmittances, the fundamental problem is to solve for the function $B_{\lambda}(T(p))$. Because there are several wavelengths at which the observations are made, the Planck function differs from one equation to another depending on the wavelength. Thus, it becomes vitally important for the direct inversion problem to eliminate the wavelength dependence in this function. In the vicinity of the 15

$\mu\text{m CO}_2$ band, it is sufficient to approximate the Planck function in a linear form as

$$B_\lambda(T(p)) = c_\lambda B_{\lambda_0}(T(p)) + d_\lambda$$

where λ_0 denotes a fixed reference wavelength and c_λ and d_λ are empirically derived constants. Assuming without loss of generality that $\tau_\lambda(p_s) = 0$, we have the following form of the RTE

$$r_\lambda = \int_{p_s}^0 b(p) W_\lambda(p) dp,$$

where

$$r_\lambda = \frac{I_\lambda - d_\lambda}{c_\lambda},$$

$$b(p) = B_{\lambda_0}(T(p)),$$

and

$$W_\lambda(p) = \frac{d\tau_\lambda(p)}{dp}$$

This is the well-known Fredholm equation of the first kind. $W_\lambda(p)$, the weighting function, is the kernel, and $b(p)$, the Planck radiance profile, is the function to be recovered from a set of observed radiances r_λ , $\lambda = 1, 2, \dots, M$, where M is the total number of spectral channels observed.

The solution of this equation is an ill-posed problem, since the unknown profile is a continuous function of pressure and there are only a finite number of observations. It is convenient to express $b(p)$ as a linear function of L variables in the form

$$b(p) = \sum_{j=1}^L b_j f_j(p),$$

where b_j are unknown coefficients, and $f_j(p)$ are the known representation functions which could be orthogonal functions, such as polynomials or Fourier series. It follows that

$$r_\lambda = \sum_{j=1}^L b_j \int_{p_s}^0 f_j(p) W_\lambda(p) dp, \quad \lambda = 1, 2, \dots, M.$$

Upon defining the known values in the form

$$H_{\lambda j} = \int_{p_s}^0 f_j(p) W_\lambda(p) dp,$$

then

$$r_\lambda = \sum_{j=1}^L H_{\lambda j} b_j, \quad \lambda = 1, 2, \dots, M.$$

In order to find b_j ($j = 1, \dots, L$), one needs to have the r_λ ($\lambda = 1, \dots, M$) where $M \geq L$. In matrix form (see Appendix A on matrices), radiances are then related to temperature

$$r = H b.$$

The solution can be written

$$b = H^{-1} r.$$

To find the solution b , the inverse matrix must be calculated.

It has been pointed out in many studies that the solution is unstable because the equation is under-constrained. Furthermore, the instability of this solution may also be traced to the following sources of error: (a) the errors arising from the numerical quadrature used for the calculation of H_{λ_j} , (b) the approximation to the Planck function, and (c) the numerical round-off errors. In addition, sounding radiometers possess inherent instrumental noise, and thus the observed radiances generate errors probably in a random fashion. All of these errors make the direct inversion from the solution of transfer equation difficult.

5.5 Linearization of the RTE

Many of the techniques for solving the RTE require linearization in which the dependence of Planck radiance on temperature is linearized, often with a first order Taylor expansion about a mean condition. Defining the mean temperature profile condition as $T_m(p)$, then

$$B_\lambda(T) = B_\lambda(T_m) + \left. \frac{\partial B_\lambda(T)}{\partial T} \right|_{T=T_m} (T - T_m)$$

and the RTE can be written

$$I_\lambda + \left. \frac{\partial B_\lambda(T)}{\partial T} \right|_{T=T_{mb\lambda}} (T_{b\lambda} - T_{mb\lambda}) = B_\lambda(T_s) \tau_\lambda(p_s) + \int_{p_s}^0 \left\{ B_\lambda(T_m) + \left. \frac{\partial B_\lambda(T)}{\partial T} \right|_{T=T_m} (T - T_m) \right\} \frac{\partial \tau_\lambda(p)}{\partial \ln p} d \ln p$$

where $T_{b\lambda}$ represents the brightness temperature for spectral band λ . Reducing to simplest form

$$(\Delta T_{b\lambda}) = \int_{p_s}^0 (\Delta T) \left(\left. \frac{\partial B_\lambda(T)}{\partial T} \right|_{T=T_m} / \left. \frac{\partial B_\lambda(T)}{\partial T} \right|_{T=T_{mb\lambda}} \right) \frac{\partial \tau_\lambda(p)}{\partial \ln p} d \ln p$$

where Δ denotes temperature difference from the mean condition. This linear form of the RTE can then be written in numerical quadrature form

$$(\Delta T_b)_\lambda = \sum_{j=1}^N W_{\lambda j} (\Delta T)_j \quad \lambda = 1, \dots, M$$

where $W_{\lambda j}$ is the obvious weighting factor, M is the number of spectral bands, and N is the number of levels at which a temperature determination is desired.

5.6 Statistical Solutions for the Inversion of the RTE

A number of methods are now presented which can be utilized to stabilize the solution and give reasonable results.

5.6.1 Statistical Least Squares Regression

Consider a statistical ensemble of simultaneously observed radiances and temperature profiles. One can define a least squares regression solution as the one that minimizes the error

$$\frac{\partial}{\partial b_k} \sum_{\lambda=1}^M \left\{ \sum_{j=1}^L H_{\lambda j} b_j - r_{\lambda} \right\}^2 = 0,$$

which leads to

$$b = (H^t H)^{-1} H^t r.$$

The least squares regression solution was used for the operational production of soundings from the very first sounding spectrometer data by Smith (1970). The form of the direct inverse solution, where r_{λ} are observations which include the measurement error, is found to be

$$b = A r$$

where A is a matrix of solution coefficients. One can define A as that matrix which gives the best least squares solution for b in a statistical ensemble of simultaneously observed radiances and temperature profiles.

The advantages of the least squares regression method over other methods are: (a) if one uses real radiance and radiosonde data comparisons to form the statistical sample, one does not require knowledge of the weighting functions or the observation errors, (b) the instrument need not be calibrated in an absolute sense, and (c) the regression is numerically stable.

Some shortcomings of the regression method are: (a) it disregards the physical properties of the RTE in that the solution is linear whereas the exact solution is non-linear because the weighting function W and consequently the solution coefficients A are functions of temperature, (b) the solution uses the same operator matrix for a range of radiances depending upon how the sample is stratified, and thus the solution coefficients are not situation dependent, and (c) radiosonde data is required, so that the satellite sounding is dependent on more than just surface data.

5.6.2 Constrained Linear Inversion of RTE

The instrument error must be taken into account. The measured radiances always contain errors due to instrument noise and biases. Therefore we write,

$$r_{\lambda}^{\text{meas}} = r_{\lambda}^{\text{true}} + e_{\lambda}$$

where e_{λ} represents the measurement errors. Thus to within the measurement error, the solution $b(p)$ is not unique. To determine the best solution, constrain the following function to be a minimum

$$\sum_{\lambda=1}^M e_{\lambda}^2 + \gamma \sum_{j=1}^L (b_j - b^{\text{mean}})^2$$

where γ is a smoothing coefficient which determines how strongly the solution is constrained to be near the mean. A least squares solution with quadratic constraints implies

$$\frac{\partial}{\partial b_k} \left[\sum_{\lambda=1}^M e_{\lambda}^2 + \gamma \sum_{j=1}^L (b_j - b^{\text{mean}})^2 \right] = 0 .$$

But

$$e_{\lambda} = \sum_{j=1}^L H_{\lambda j} b_j - r_{\lambda}^{\text{true}}$$

which leads to

$$\sum_{\lambda=1}^M \left[\sum_{j=1}^L H_{\lambda j} b_j - r_{\lambda}^{\text{true}} \right] H_{\lambda k} + \gamma [b_k - b^{\text{mean}}] = 0 .$$

By definition

$$b^{\text{mean}} = \frac{1}{L} \sum_{j=1}^L b_j ,$$

and

$$b_k - b^{\text{mean}} = -L^{-1} b_1 - L^{-1} b_2 - \dots + (1-L^{-1}) b_k - \dots - L^{-1} b_L .$$

So the constrained least squares solution can be written in matrix form

$$H^t H b - H^t r + \gamma M b = 0 ,$$

where

$$M = \begin{matrix} 1-L^{-1} & -L^{-1} & . & . & . \\ -L^{-1} & 1-L^{-1} & & & \\ -L^{-1} & -L^{-1} & 1-L^{-1} & & \\ -L^{-1} & -L^{-1} & -L^{-1} & 1-L^{-1} & \\ . & . & . & . & \\ . & . & . & . & 1-L^{-1} \end{matrix}$$

which becomes the identity matrix as L approaches ∞ . Thus the solution has the form

$$b = (H^t H + \gamma M)^{-1} H^t r .$$

This is the equation for the constrained linear inversion derived by Phillips (1962) and Twomey (1963). We will discuss this further in the section on the Minimum Information Solution.

5.6.3 Statistical Regularization

To make explicit use of the physics of the RTE in the statistical method, using the linearized form of the RTE, one can express the brightness temperatures for the statistical ensemble of profiles as

$$T_b = TW + E ,$$

where E is a matrix of the unknown observational errors. The temperature difference notation has been dropped for simplicity. Solving with the least squares approach, as explained earlier, yields

$$A = (W^t T^t T W + E^t E)^{-1} W^t T^t T,$$

where covariances between observation error and temperature ($E^t T$) are assumed to be zero since they are uncorrelated. Defining the covariance matrices

$$S_T = \frac{1}{S-1} (T^t T) \text{ and } S_E = \frac{1}{S-1} (E^t E)$$

where S indicates the size of the statistical sample; then

$$A = (W^t S_T W + S_E)^{-1} W^t S_T.$$

The solution for the temperature profile is

$$T = T_b (W^t S_T W + S_E)^{-1} W^t S_T.$$

This solution was developed independently by Strand and Westwater (1968), Rodgers (1968), and Turchin and Nozik (1969).

The objections raised about the regression method do not apply to this statistical regularization solution, namely: (a) W is included and its temperature dependence can be taken into account through iteration; (b) the solution coefficients are re-established for each new temperature profile retrieval; and (c) there is no need for coincident radiosonde and satellite observations so that one can use an historical sample to define S_T .

The advantages of the regression method are, however, the disadvantages of the statistical regularization method, namely: (a) the weighting functions must be known with higher precision; and (b) the instrument must be calibrated accurately in an absolute sense.

As with regression, the statistical regularization solution is stable because S_T and S_E are strongly diagonal matrices which makes the matrix

$$(S^t S_T W + S_E)$$

well-conditioned for inversion.

5.6.4 Minimum Information Solution

Twomey (1963) developed a temperature profile solution to the radiances that represents a minimal perturbation of a guess condition such as a forecast profile. In this case T represents deviations of the actual profile from the guess and T_b represents the deviation of the observed brightness temperatures from those which would have arisen from the guess profile condition. S_T is then a covariance matrix of the errors in the guess profile, which is unknown. Assume that the errors in the guess are uncorrelated from level to level such that

$$S_T = \sigma_T^2 I$$

where I is the identity matrix and σ_T^2 is the expected variance of the errors in the guess. If one also assumes that the measurement errors are random, then

$$S_E = \sigma_E^2 I.$$

Simplifying the earlier expression for a solution using statistical regularization, we get

$$T = T_b (W^t W + \gamma I)^{-1} W^t$$

where

$$\gamma = \sigma_\epsilon^2 / \sigma_T^2 (\approx 10^{-3}).$$

The solution given is the Tikhonov (1963) method of regularization.

The solution is generally called the Minimum Information Solution since it requires only an estimate of the expected error of the guess profile. One complication of this solution is that γ is unknown. However, one can guess at γ (e.g., 10^{-3}) and iterate it until the solution converges

$$\frac{1}{M} \sum_{j=1}^M (T_{bi} - T_{bj})^2 \leq \sigma_\epsilon^2.$$

The minimum information solution was used for processing sounding data by the SIRS-B and VTPR instruments.

5.6.5 Empirical Orthogonal Functions

It is often advantageous to expand the temperature profile for the N pressure levels so that

$$T(p_j) = \sum_{k=1}^L a_k f_k(p_j) \quad j = 1, \dots, N$$

where L is the number of basis functions (less than M the number of spectral bands) and $f_k(p_j)$ are some type of basis functions (polynomials, weighting functions, or empirical orthogonal functions).

An empirically optimal approximation is achieved by defining $f_k(p_j)$ as empirical orthogonal functions (EOF) which are the eigenvectors of a statistical covariance matrix of temperature $T^t T$. When the eigenvectors and associated eigenvalues of $(T^t T)$ are determined and the N eigenvalues are ordered from largest to smallest, the associated eigenvectors will be ordered according to the amount of variance they explain in the empirical sample of soundings used to determine $T^t T$. The EOF's are optimal basis functions in that the first EOF $f_1(p_j)$ is the best single predictor of $T(p)$ that can be found in a mean squared error sense to describe the values used to form $T^t T$. The second EOF is the best prediction of the variance unexplained by $f_1(p_j)$, and so on. Wark and Fleming (1966) first used the EOF approximation in the linear RTE.

The eigenvectors of the temperature covariance matrix (empirical orthogonal functions) provide the most economical representation of a large sample of observations, where each observation consists of a set of numbers which are not statistically independent of each other. Each observation can be represented as a linear combination of functions (vectors) so that the coefficients in the representation are statistically independent. These functions, which are the eigenvectors of the statistical covariance matrix, are the optimum descriptors in the sense that the progressive explanation of variance is maximized. In other words, among all possible sets of orthogonal functions of a physical variable, the first n empirical functions explain more variance than the first n functions of any other set.

To see this more clearly, consider the representation of the temperature T_{is} at level I from sample observation s. The covariance matrix of the atmospheric profile is given by

$$\hat{T}_{ij} = \frac{1}{S} \sum_{s=1}^S (T_{is} - \bar{T}_i) (T_{js} - \bar{T}_j),$$

where without loss of generality we declare the mean temperatures for each level to be zero, so that

$$\hat{T}_{ij} = \frac{1}{S} \sum_{s=1}^S T_{is} T_{js}$$

or

$$\hat{T} = T^t T.$$

This represents an NxN matrix. If we consider that T is an NxS matrix (S measurements at N levels),

$$\begin{array}{cccc} T_{11}, & T_{12}, & \dots, & T_{1S} \\ T_{21}, & T_{22}, & \dots, & T_{2S} \\ T_{N1} & T_{N2}, & \dots, & T_{NS} \end{array}$$

then \hat{T} is the product of an SxN (T^t) and an NxS (T) matrices. It should be noted that \hat{T}_{ij} is the covariance of temperature at levels i and j and is zero only if temperatures at these two levels are uncorrelated. The diagonal element \hat{T}_{kk} is the variance of the atmospheric temperature at the level

We diagonalize \hat{T} by performing an eigenvalue analysis. We write

$$\hat{T}E = E\Lambda$$

where E is matrix of eigenvector columns and Λ is diagonal matrix of eigenvalues. In expanded notation, the eigenvalue problem can be stated

$$\hat{T} \begin{array}{c} E_{1i} \\ E_{2i} \\ \vdots \\ E_{Ni} \end{array} = \lambda_i \begin{array}{c} E_{1i} \\ E_{2i} \\ \vdots \\ E_{Ni} \end{array}.$$

in matrix notation

$$\hat{T} \begin{array}{ccc} E_{11} & E_{12} \dots E_{1N} \\ E_{21} & E_{22} & E_{2N} \\ \vdots & \vdots & \vdots \\ E_{N1} & E_{N2} & E_{NN} \end{array} = \begin{array}{ccc} E_{11} & E_{12} \dots E_{1N} & \lambda_1 \\ E_{21} & E_{22} & E_{2N} & \lambda_2 \\ \vdots & \vdots & \vdots & \vdots \\ E_{N1} & E_{N2} & E_{NN} & \lambda_N \end{array}.$$

where

$$E = \begin{array}{ccc} E_1 & E_2 & \dots & E_N \\ \downarrow & \downarrow & & \downarrow \end{array}$$

\hat{T}

Since T is real and symmetric, it is Hermitian and therefore has eigenvectors that are orthonormal and eigenvalues that are real and greater than zero. Thus E is an orthogonal matrix

$$E^t E = I \quad \text{or} \quad E^t = E^{-1} .$$

The eigenvectors form a basis for the temperature variances. Any temperature variance can be expressed as an expansion of these EOFs.

The transformation that diagonalizes \hat{T} emerges

$$E^t \hat{T} E = \Lambda \quad \text{or} \quad \hat{T} = E \Lambda E^t .$$

When the square root of the eigenvalue of the temperature covariance matrix is less than the accuracy of the temperature measurements, its contribution to the solution of the temperature profile is unreliable (it is merely fitting noise). The eigenvectors are ordered in such a way that the first eigenvector explains largest amount of variance, describing largest scale of variability, and subsequent eigenvectors account for the residual variance in successively decreasing order. The first few eigenvectors account for all significant variance, the remaining eigenvectors are merely fitting noise. This suggests a desired representation of each profile

$$T_{is} = \sum_{k=1}^L A_{ks} E_{ik} \quad \text{or} \quad T = E A$$

(NxS) (LxS) (NxL) (NxS) (NxL) (LxS)

where L is the number of EOFs associated with eigenvalues whose square root is greater than the noise. The sample of coefficients a_{ks} are statistically independent, hence

$$\frac{1}{S} \sum_{s=1}^S A_{is} A_{js} = \lambda_i \delta_{ij} \quad \text{or} \quad \frac{1}{S} A^t A = \Lambda .$$

Therefore, the temperature profile retrieval from empirical orthogonal functions has N equal to 25 levels, M equal to 18 channels, L equal to 10 EOF, and S equal to the sample of 1200. Using the convention that capital letters denote matrices and lower case letters denote vectors in the following paragraphs, we can write the expansion of atmospheric temperature in terms of EOF as

$$t = E a$$

(Nx1) (NxL) (Lx1)

where the solution rests in finding the expansion coefficients a , which are dependent on the atmospheric situation. Also the observed brightness temperatures can be expanded in terms of the EOF for the brightness temperature covariance matrix, so that

$$t_b = E_b a_b ,$$

(Mx1) (MxL) (Lx1)

where all components of this equation are known. Note that in expanded form

$$\begin{array}{ccccccc} T_{1s} & & E_{11} & E_{21} & \dots & E_{10,1} & A_{1s} \\ T_{2s} & & E_{12} & E_{22} & & & A_{2s} \\ \cdot & = & \cdot & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot & \\ \cdot & & \cdot & \cdot & & \cdot & \end{array}$$

$$T_{25S} \quad E_{1,25} \quad E_{10,25} \quad A_{10s}$$

which is different from $a^t E^t$

$$A_{1s}, A_{2s}, \dots, A_{10s} \quad \begin{array}{l} E_{11} \quad E_{21} \quad \dots \quad E_{25,1} \\ E_{12} \quad E_{22} \\ \cdot \\ \cdot \\ \cdot \\ E_{1,10} \quad \quad E_{25,10} \end{array}$$

We are trying to solve for t from t_b ; a more stable solution occurs when an intermediate step is inserted to get a from a_b . In this formulation a transformation matrix D is used. Then

$$\begin{array}{l} a \\ (L \times 1) \end{array} = D \begin{array}{l} a_b \\ (L \times L) \end{array} = D (E_b^t E_b)^{-1} E_b^t t_b = D E_b^t t_b$$

where the least squares solution has been inferred and the orthogonality property has been used. D is best determined from a statistical sample of 1200 radiosonde and rocketsonde profiles covering all seasons of the year throughout both hemispheres. So we write

$$\begin{array}{l} A \\ (L \times S) \end{array} = D \begin{array}{l} A_b \\ (L \times L)(L \times S) \end{array}$$

then least squares solution for D yields

$$D = A A_b^t (A_b A_b^t)^{-1}.$$

Using the 1200 samples we have

$$\begin{array}{l} T \\ (N \times S) \end{array} = E \begin{array}{l} A \\ (N \times L)(L \times S) \end{array}$$

which implies that the least squares solution for A yields

$$A = (E^t E)^{-1} E^t T = E^t T.$$

since by orthogonality $E^t E = 1$. Similarly for the brightness temperature terms

$$\begin{array}{l} T_b \\ (M \times S) \end{array} = E_b \begin{array}{l} A_b \\ (M \times L)(L \times S) \end{array}$$

where a least squares solution for A_b gives

$$A_b = (E_b^t E_b)^{-1} E_b^t T_b = E_b^t T_b.$$

Through the solutions for A and A_b , D is known,

$$\begin{aligned} D &= E^t T [E_b^t T_b]^t [(E_b^t T_b)(E_b^t T_b)^t]^{-1} \\ &= E^t T T_b^t E_b [E_b^t T_b T_b^t E_b]^{-1} \\ &= E^t T T_b^t E_b E_b^{-1} T_b^{-1} T_b^{-1} E_b, \end{aligned}$$

or

$$D = E^t \quad T \quad T_b^{-1} \quad E_b.$$

$$(L \times L) = (L \times N) (N \times S) (S \times M) (M \times L)$$

Solving for t

$$\begin{aligned} t &= E a = E D a_b = E D E_b^t t_b = H t_b \\ &= E E^t T T_b^{-1} E_b E_b^t t_b \end{aligned}$$

which becomes

$$\begin{matrix} t &= & T & T_b^{-1} & t_b \\ (N \times 1) & & (N \times S) & (S \times M) & (M \times 1) \end{matrix}$$

The ordinary least squares solution yields

$$t = (T T_b^t) (T_b T_b^t)^{-1} t_b .$$

The advantage of eigenvector approach is that it is less sensitive to instrument noise (low eigenvalue eigenvectors have been discarded). But if all eigenvectors are used ($L=M$) then the EOF solution is same as the least squares solution. It is better conditioned because $L < M$ and noise has not been fit, but true variance has been. The advantages of regression are: (1) you don't need to know the weighting functions or the measurement errors, (2) instrument calibration is not critical, and (3) the regression is numerically stable. The disadvantages are: (1) there is no physics of the RTE included, (2) there is a linear assumption, (3) sample stratification is crucial, and (4) it is dependent on radiosonde data.

In practice, the empirical function series is truncated either on the basis of the smallness of the eigenvalues (thus, the smallness of explained variance) of higher order eigenvectors or on the basis of numerical instabilities which result when L approaches M . If $L \leq M$ and L is small (e.g., ≤ 5), a stable solution can usually be obtained by the direct inverse. The matrix H , in this case, is better conditioned with respect to matrix inversion. This is because the basis vector f_k is smooth and acts as a constraint on the solution thereby stabilizing it. However, in practice, best results are obtained by choosing an optimum $L < M$ or by conditioning the H matrix prior to its inversion.

5.7 Numerical Solutions for the Inversion of the RTE

We have discussed several statistical matrix solutions of the direct linear inversion of the RTE; we now turn our attention to numerical iterative techniques producing solutions.

5.7.1 Numerical Iteration Solution by Chahine Relaxation Method

The difficulty in reconstructing the temperature profile from radiances at several selected wavelengths is due to the fact that the Fredholm equation with fixed limits may not always have a solution for an arbitrary function. Since the radiances are obtained from measurements which are only approximate, the reduction of this problem to a linear system is mathematically improper, and a nonlinear approach to the solution of the full radiative transfer equations appears to become necessary. The basic radiance equation is:

$$I_\lambda = B_\lambda(T_s) \tau_\lambda(p_s) + \int_{p_s}^0 B_\lambda(T(p)) \frac{d\tau_\lambda(p)}{d \ln p} d \ln p, \quad \lambda = 1, 2, \dots, M,$$

where λ denotes the different spectral channels and the weighting function is expressed in logarithmic scale. Since the weighting function reaches a strong maximum at different pressure levels for different spectral channels, the actual upwelling radiance observed by the satellite, R_λ , can be approximated through the use of the mean value theorem, by

$$R_\lambda = B_\lambda(T_s) \tau_\lambda(p_s) + B_\lambda(T(p_\lambda)) \left[\frac{d\tau_\lambda(p)}{d \ln p} \right] \Big|_{p_\lambda} \Delta_\lambda \ln p ,$$

where p_λ denotes the pressure level at which the maximum weighting function is located, and $\Delta_\lambda \ln p$ is the differential of the pressure at the λ th level and is defined as the effective width of the weighting function for wavelength λ . Let the guessed temperature at p_λ level be $T'(p_\lambda)$. Thus, the guessed upwelling radiance I_λ is given by:

$$I_\lambda = B_\lambda(T_s) \tau_\lambda(p_s) + B_\lambda(T'(p_\lambda)) \left[\frac{d\tau_\lambda(p)}{d \ln p} \right] \Big|_{p_\lambda} \Delta_\lambda \ln p ,$$

where the transmittance and the surface temperature are assumed to be known.

Upon dividing and noting that the dependence of the Planck function on temperature variations is much stronger than that of the weighting function, we obtain

$$\frac{R_\lambda - B_\lambda(T_s) \tau_\lambda(p_s)}{I_\lambda - B_\lambda(T_s) \tau_\lambda(p_s)} \approx \frac{B_\lambda(T(p_\lambda))}{B_\lambda(T'(p_\lambda))}$$

When the surface contribution to the upwelling radiance is negligible or dominant, the equation may be approximated by

$$\frac{R_\lambda}{I_\lambda} \approx \frac{B_\lambda(T(p_\lambda))}{B_\lambda(T'(p_\lambda))}$$

or in iteration form

$$\frac{R_\lambda}{I_\lambda^{\text{old}}} = \frac{B_\lambda(T^{\text{new}}(p_\lambda))}{B_\lambda(T^{\text{old}}(p_\lambda))} .$$

This is the relaxation equation developed by Chahine (1970).

Since most of the upwelling radiance at the strong absorption bands arises from the upper parts of the atmosphere, whereas the radiance from the less attenuating bands comes from progressively lower levels, it is possible to select a set of wave numbers to recover the atmospheric temperature at different pressure levels. The size of a set of sounding wave numbers is defined by the degree of the vertical resolution required and is obviously limited by the capacity of the sounding instrument.

Assuming now that the upwelling radiance is measured at a discrete set of M spectral channels, and that the composition of carbon dioxide and the level of the weighting function peaks p_λ are all known, the following integration procedures are utilized to recover the temperature profile $T^{(n)}(p_\lambda)$ at level p_λ , where n is the order of the iterations:

- (a) Make an initial guess for $T^{(n)}(p_\lambda)$, $n = 0$;
- (b) Substitute $T^{(n)}(p_\lambda)$ into the RTE and use an accurate quadrature formula to evaluate the expected upwelling radiance $I_\lambda^{(n)}$ for each sounding channel;

- (c) Compare the computed radiance values $I_{\lambda}^{(n)}$ with the measured data R_{λ} . If the residuals

$$[R_{\lambda} - I_{\lambda}^{(n)}] / R_{\lambda}$$

are less than a preset small value (say, 10^{-4}) for each sounding channel, then $T^{(n)}(p_{\lambda})$ is a solution;

- (d) If the residuals are greater than the preset criterion, we apply the relaxation equation to each wavelength (M times) to generate a new guess for the temperature values

$$T^{(n+1)}(p_{\lambda})$$

at the selected pressure levels p_{λ} . Note that

$$T^{(n+1)}(p_{\lambda}) = B^{-1} \left[B(T^{(n)}(p_{\lambda})) \frac{R_{\lambda}}{I_{\lambda}^{(n)}} \right].$$

In this calculation, each sounding channel acts at only one specific pressure level p_{λ} to relax

$$T^{(n)}(p_{\lambda}) \text{ to } T^{(n+1)}(p_{\lambda});$$

- (e) Carry out the interpolation between the temperature value at each given level p_{λ} to obtain the desirable profile (it is sufficient to use linear interpolation);
- (f) Finally, with this new temperature profile, go back to step (b) and repeat until the residuals are less than the preset criterion.

5.7.2 Example Problem Using the Chahine Relaxation Method

Consider a three channel radiometer with spectral bands centred at 676.7, 708.7, and 746.7 wavenumbers. Their weighting functions peak at 50, 400, and 900 mb, respectively. The transmittance is summarized in the following table:

Pressure (mb)	Transmittance		
	676.7	708.7	746.7 (cm^{-1})
10	.86	.96	.98
150	.05	.65	.87
600	.00	.09	.61
1000	.00	.00	.21

The surface temperature is assumed to be 280 K. The radiometer senses the radiances R_i for each spectral band i to be 45.2, 56.5, and 77.8 $\text{mW/m}^2/\text{ster/cm}^{-1}$, respectively.

- (a) Guess $T^{(0)}(50) = T^{(0)}(400) = T^{(0)}(900) = 260 \text{ K}$;
- (b) Compute the radiance values for this guess profile by writing:

$$I_i^{(0)} = B_i(1000) \tau_i(1000) + B_i(900) (\tau_i(600) - \tau_i(1000)) \\ + B_i(400) (\tau_i(150) - \tau_i(600))$$

$$+ B_i(50) (\tau_i(10) - \tau_i(150))$$

yielding 76.9, 82.3, and 85.2 mW/m²/ster/cm⁻¹, respectively;

- (c) Convergence has not been reached;
 (d) Iterate to a new profile using the relaxation equation

$$T^{(1)}(p_i) = B_i^{-1} \left[B(T^{(0)}(p_i)) \frac{R_i}{I_i^{(0)}} \right]$$

yielding 228, 238, and 254 K, respectively.

- (e) Disregard interpolation of temperature to other pressure levels in this example and go back to (b).

(b') 45.7, 55.3, 71.6 mW/m²/ster/cm⁻¹

(c') no convergence

(d') 228, 239, 259 K

(b'') 45.3, 56.4, 74.4 mW/m²/ster/cm⁻¹

(c'') no convergence

(d'') 228, 239, 262 K

(b''') 45.2, 56.7, 75.9 mW/m²/ster/cm⁻¹

(c''') no convergence

(d''') 228, 239, 264 K

(b''') 45.2, 56.8, 76.7 mW/m²/ster/cm⁻¹

(c''') convergence within 1 mW/m²/ster/cm⁻¹

Thus, the temperature retrieval yields $T(50) = 228$ K, $T(400) = 239$ K, and $T(900) = 264$ K.

5.7.3 Smith's Numerical Iteration Solution

Smith (1970) developed an iterative solution for the temperature profile retrieval, which differs somewhat from that of the relaxation method introduced by Chahine. As before, let R_λ denote the observed radiance and $I_\lambda^{(n)}$ the computed radiance in the n th iteration. Then the upwelling radiance expression may be written as:

$$I_\lambda^{(n)} = B_\lambda^{(n)}(T_s) \tau_\lambda(p_s) + \int_{p_s}^0 B_\lambda^{(n)}(T(p)) \frac{d\tau_\lambda(p)}{d \ln p} d \ln p.$$

Further, for the $(n+1)$ step we set

$$\begin{aligned} R_\lambda &= I_\lambda^{(n+1)} \\ &= B_\lambda^{(n+1)}(T_s) \tau_\lambda(p_s) + \int_{p_s}^0 B_\lambda^{(n+1)}(T(p)) \frac{d\tau_\lambda(p)}{d \ln p} d \ln p. \end{aligned}$$

Upon subtracting, we obtain

$$R_\lambda - I_\lambda^{(n)} = [B_\lambda^{(n+1)}(T_s) - B_\lambda^{(n)}(T_s)] \tau_\lambda(p_s) + \int_{p_s}^0 [B_\lambda^{(n+1)}(T(p)) - B_\lambda^{(n)}(T(p))] \frac{d\tau_\lambda(p)}{d \ln p} d \ln p$$

An assumption is made at this point that for each sounding wavelength, the Planck function difference for the sensed atmospheric layer is independent of the pressure coordinate. Thus,

$$R_\lambda - I_\lambda^{(n)} = B_\lambda^{(n+1)}(T(p)) - B_\lambda^{(n)}(T(p)).$$

That is,

$$B_\lambda^{(n+1)}(T(p)) = B_\lambda^{(n)}(T(p)) + (R_\lambda - I_\lambda^{(n)}).$$

This is the iteration equation developed by Smith. Moreover, for each wavelength we have

$$T_\lambda^{(n+1)}(p) = B_\lambda^{-1}[B_\lambda(T^{(n+1)}(p))].$$

Since the temperature inversion problem now depends on the sounding wavelength λ , the best approximation of the true temperature at any level p would be given by a weighted mean of independent estimates so that

$$T^{(n+1)}(p) = \frac{\sum_{\lambda=1}^M T_\lambda^{(n+1)}(p) W_\lambda(p)}{\sum_{\lambda=1}^M W_\lambda(p)},$$

where the proper weights should be approximately

$$W_\lambda(p) = \left\{ \begin{array}{l} d\tau_\lambda(p), \quad p < p_s \\ \tau_\lambda(p), \quad p = p_s \end{array} \right\}.$$

It should be noted that the numerical technique presented above makes no assumption about the analytical form of the profile imposed by the number of radiance observations available. The following iteration schemes for the temperature retrieval may now be employed:

- (a) Make an initial guess for $T^{(n)}(p)$, $n = 0$;
- (b) Compute $B_\lambda^{(n)}(T(p))$ and $I_\lambda^{(n)}$;
- (c) Compute $B_\lambda^{(n+1)}(T(p))$ and $T_\lambda^{(n+1)}(p)$ for the desired levels;
- (d) Make a new estimate of $T^{(n+1)}(p)$ using the proper weights;
- (e) Compare the computed radiance values $I_\lambda^{(n)}$ with the measured data R_λ . If the residuals

$$\Delta^{(n)} = |R_\lambda - I_\lambda^{(n)}| / R_\lambda.$$

are less than a preset small value, then $T^{(n+1)}(p)$ would be the solution. If not, repeat steps (b)-(d) until convergence is achieved.

5.7.4 Example Problem Using Smith's Iteration

Using the data from the three channel radiometer discussed in the previous example

involving the relaxation method, we proceed as before.

- (a) Guess $T^{(0)}(50) = T^{(0)}(400) = T^{(0)}(900) = 260$ K;
- (b) Compute the estimated radiance values as before giving 76.9, 82.3, 85.2 mW/m²/ster/cm⁻¹ for $I_i^{(0)}$;
- (c) For each spectral band I , calculate a new profile from

$$T_i^{(1)}(p_j) = B_i^{-1} \{ B(T^{(0)}(p_j)) + (R_i - I_i^{(0)}) \}$$

where j runs over all desired pressure levels. This yields

$$233, 233, 233 \text{ K for } T_1^{(1)}, \text{ and}$$

$$239, 239, 239 \text{ K for } T_2^{(1)}, \text{ and}$$

$$254, 254, 254 \text{ K for } T_3^{(1)}.$$

- (d) The next iteration profile will be given by the weighted mean

$$T^{(1)}(p_j) = \frac{\sum_{i=1}^3 T_i^{(1)}(p_j) \Delta\tau_i(p_j)}{\sum_{i=1}^3 \Delta\tau_i(p_j)}$$

which yields 237, 243, 251 K.

- (e) No convergence yet, using the arbitrary criterion that:

$$| R_i - I_i | < 1 \text{ mW/m}^2\text{/ster/cm}^{-1}.$$

- (b') 52.9, 60.8, 72.5 mW/m²/ster/cm⁻¹ are $I_i^{(1)}$.

- (c') $T_1^{(2)}$ is 229, 236, 245 K

$$T_2^{(2)} \text{ is } 232, 239, 248 \text{ K}$$

$$T_3^{(2)} \text{ is } 242, 248, 256 \text{ K}$$

- (d') $T^{(2)}$ is 231, 241, 254 K

- (e') No convergence yet.

- (b'') 48.2, 58.4, 72.8 mW/m²/ster/cm⁻¹ are $I_i^{(2)}$

- (c'') $T_1^{(3)}$ is 228, 239, 252 K

$$T_2^{(3)} \text{ is } 229, 240, 253 \text{ K}$$

$$T_3^{(3)} \text{ is } 236, 246, 258 \text{ K}$$

- (d'') $T^{(3)}$ is 229, 241, 257 K

- (e'') No convergence yet.

- (b''') 46.5, 58.2, 74.1 mW/m²/ster/cm⁻¹ are $I_i^{(4)}$

- (c''') $T_1^{(4)}$ is 228, 240, 256 K
 $T_2^{(4)}$ is 227, 240, 256 K
 $T_3^{(4)}$ is 233, 245, 260 K
- (d''') $T^{(4)}$ is 228, 241, 259 K
- (e''') Convergence in next iteration.
- (b''') 45.7, 58.1, 75.1 mW/m²/ster/cm⁻¹ are $I_i^{(4)}$
 which are within 1 mW/m²/ster/cm⁻¹ are $I_i^{(3)}$.
- (c''') $T_1^{(5)}$ is 228, 241, 259 K
 $T_2^{(5)}$ is 226, 240, 258 K
 $T_o^{(5)}$ is 231, 244, 261 K
- (d''') $T^{(5)}$ is 228, 241, 261 K

Thus the temperature retrieval yields $T(50) = 228$ K, $T(400) = 241$ K, and $T(900) = 261$ K. This result compares reasonably well with the earlier result obtained by the relaxation method.

5.7.5 Comparison of the Chahine and Smith Numerical Iteration Solution

Figure 5.4 illustrates a retrieval exercise using both Chahine's and Smith's methods. The same transmittances were used and the true temperature profile is shown. A climatological profile was used as an initial guess, and the surface temperature was fixed at 279.5 K. The observed radiances utilized were obtained by direct computations for six VTPR channels at 669.0, 676.7, 694.7, 708.7, 723.6, and 746.7 cm⁻¹ using a forward difference scheme. Numerical procedures already outlined were followed, and a linear interpolation with respect to $\ln p$ was used in the relaxation method to get the new profile. With the residual set at 1%, the relaxation method converged after six iterations, and results are given by the solid line with black dots. Since the top level at which the temperature was calculated was about 20 mb, extrapolation to the level of 1 mb was used. Recovered results using Smith's method are displayed by the dashed line. No interpolation is necessary since this method gives temperature values at desirable levels. It took five iterations to converge the solution to within 1%. Both methods do not adequately recover the temperature at upper levels due to the fact that the highest weighting function peak is at about 30 mb. It should be noted that the retrieval exercise presented here does not account for random errors and therefore, it is a hypothetical one.

The major problems with the Chahine method are: (a) the profile is not usually well-represented by a series of line segments between pressure levels where the weighting functions peak, particularly for a small number of channels (levels), and (b) the iteration and hence the solution can become unstable since one is attempting to extract M distinct pieces of information from M non-independent observations.

While the Smith method does avoid the problems of the Chahine method (no interpolation is required for a temperature at any pressure level and the solution is stable in the averaging scheme because the random error propagating from R_λ to $T(p)$ is suppressed to the average value of the errors in all channels, which will be near zero), it does have the main disadvantage that the averaging process can prevent obtaining a solution that satisfies the observations to within their measurement error levels. There is no guarantee that the solution converges to one which satisfies

the radiances by this criterion.

5.8 Direct Physical Solution

5.8.1 Example Problem Solving Linear RTE Directly

The linear form of the RTE can be solved directly (often with rather poor results). For the example problem presented earlier, we have that T_b equals 223, 232, and 258 K for the spectral bands, respectively. As before, take $T(1000) = 280$ K and assume a mean temperature profile condition $T(900) = T(400) = T(50) = 260$ K. Therefore, T_b equals 250, 258, and 263 K, respectively. We set up the matrix solution by writing

$$\begin{aligned} \Delta T_{bi} = & \Delta T_{900} \left[\frac{\partial B_i}{\partial T} \bigg|_{T_{900}} / \frac{\partial B_i}{\partial T} \bigg|_{T_{bi}} \right] (\tau_i(600) - \tau_i(1000)) \\ & + \Delta T_{400} \left[\frac{\partial B_i}{\partial T} \bigg|_{T_{400}} / \frac{\partial B_i}{\partial T} \bigg|_{T_{bi}} \right] (\tau_i(150) - \tau_i(600)) \\ & + \Delta T_{50} \left[\frac{\partial B_i}{\partial T} \bigg|_{T_{50}} / \frac{\partial B_i}{\partial T} \bigg|_{T_{bi}} \right] (\tau_i(10) - \tau_i(150)) \end{aligned}$$

which gives

$$-27 = \Delta T_{900}(.89/.77)(.00) + \Delta T_{400}(.89/.77)(.05) + \Delta T_{50}(.89/.77)(.81)$$

$$-26 = \Delta T_{900}(.86/.83)(.09) + \Delta T_{400}(.86/.83)(.56) + \Delta T_{50}(.86/.83)(.31)$$

$$-5 = \Delta T_{900}(.81/.85)(.40) + \Delta T_{400}(.81/.85)(.26) + \Delta T_{50}(.81/.85)(.11)$$

Solving we find that

$$\Delta T_{900} = 15 \text{ K,}$$

$$\Delta T_{400} = -33 \text{ K,}$$

$$\Delta T_{50} = -25 \text{ K,}$$

so that the temperature profile solution is

$$T(900) = 275 \text{ K,}$$

$$T(400) = 227 \text{ K,}$$

$$T(50) = 235 \text{ K.}$$

Obviously, this example was ill-conditioned since Taylor expansion of differences larger

than 10 K is foolhardy. However, this does demonstrate how to set up a direct solution, which should be representative of the mean temperature condition used in the expansion is close to the actual temperature profile.

Typically, the direct solution is unstable because there are the unknown observation errors and W is nearly singular due to strong overlapping of the weighting functions. Since W is ill-conditioned with respect to matrix inversion, the elements of the inverse matrix are greatly inflated which, in turn, greatly amplifies the experimental error of the observations. This renders the solution virtually useless. The ill-conditioned solution results since one does not have N independent pieces of information about T from M radiation observations. The solution is further complicated because M is usually much smaller than the number of temperature points, N , needed to represent the temperature profile.

5.8.2 Simultaneous Direct Physical Solution of the RTE for Temperature and Moisture

Solution of the RTE often involves several iterations between solving for the temperature and moisture profiles. As pointed out earlier, they are interrelated but most solutions only solve for each one separately, assuming the other is known. Recently Smith (1985) has developed a simultaneous direct physical solution of both.

In order to solve for the temperature and moisture profiles simultaneously, a simplified form of the integral of the radiative transfer equation is considered,

$$R = B_o + \int_o^{p_s} \tau dB$$

which comes integrating the atmospheric term by parts in the more familiar form of the RTE. R represents the radiance, τ the transmittance, and B the Planck radiance. Dependency on angle, pressure, and frequency are neglected for simplicity. The subscript s refers to the surface level and o refers to the top of the atmosphere. Then in perturbation form, where δ represents a perturbation with respect to an a priori condition

$$\delta R = \int_o^{p_s} (\delta\tau) dB + \int_o^{p_s} \tau d(\delta B) .$$

Integrating the second term on right side of the equation by parts,

$$\int_o^{p_s} \tau d(\delta B) = \tau \delta B \Big|_o^{p_s} - \int_o^{p_s} \delta B d\tau = \tau_s \delta B_s - \int_o^{p_s} \delta B d\tau ,$$

yields

$$\delta R = \int_o^{p_s} (\delta\tau) dB + \tau_s \delta B_s - \int_o^{p_s} \delta B d\tau$$

Now writing the differentials with respect to temperature

$$\delta R = \delta T_b \frac{\partial B}{\partial T_b}, \quad \delta B = \delta T \frac{\partial B}{\partial T}$$

and with respect to pressure

$$dB = \frac{\partial B}{\partial T} \frac{\partial T}{\partial p} dp, \quad d\tau = \frac{\partial \tau}{\partial p} dp,$$

yields

$$\delta T_b = \int_0^{p_s} \delta \tau \frac{\partial T}{\partial p} \left[\frac{\partial B}{\partial T} / \frac{\partial B}{\partial T_b} \right] dp - \int_0^{p_s} \delta T \frac{\partial \tau}{\partial p} \left[\frac{\partial B}{\partial T} / \frac{\partial B}{\partial T_b} \right] dp$$

$$+ \delta T_s \left[\frac{\partial B_s}{\partial T_s} / \frac{\partial B}{\partial T_b} \right] \tau_s$$

where T_b is the brightness temperature. Finally, assume that the transmittance perturbation is dependent only on the uncertainty in the column of precipitable water density weighted path length u according to the relation

$$\delta \tau = \frac{\partial \tau}{\partial u} \delta u .$$

Thus

$$\delta T_b = \int_0^{p_s} \delta u \frac{\partial T}{\partial p} \frac{\partial \tau}{\partial u} \left[\frac{\partial B}{\partial T} / \frac{\partial B}{\partial T_b} \right] dp - \int_0^p \delta T \frac{\partial \tau}{\partial p} \left[\frac{\partial B}{\partial T} / \frac{\partial B}{\partial T_b} \right] dp$$

$$+ \delta T_s \left[\frac{\partial B_s}{\partial T_s} / \frac{\partial B}{\partial T_b} \right] \tau_s$$

$$= f [\delta u, \delta T, \delta T_s]$$

where f represents a functional relationship.

The perturbations are with respect to some a priori condition which may be estimated from climatology, regression, or more commonly from an analysis or forecast provided by a numerical model. In order to solve for δu , δT , and δT_s from a set spectrally independent radiance observations δT_b , the perturbation profiles are represented in terms of arbitrary basis functions $\varphi(p)$; so

$$\delta T_s = \alpha_0 \varphi_0$$

$$\delta u(p) = \sum_{i=1}^Q \alpha_i \int_0^p q(p) \varphi_i(p) dp ,$$

where the water vapour mixing ratio is given by $q(p) = g \partial u / \partial p$ and $\delta q = g \sum \alpha_i \varphi_i$,

$$\delta T(p) = - \sum_{i=Q+1}^L \alpha_i \varphi_i(p) .$$

Then for M spectral channel observations

$$\delta T_{bj} = \sum_{i=0}^L \alpha_i \psi_{ij} \quad \text{where } j = 1, \dots, M$$

and

$$\psi_{0j} = \left[\frac{\partial B_j}{\partial T_s} / \frac{\partial B_j}{\partial T_{bj}} \right] \tau_{sj}$$

$$\psi_{ij} = \int_0^{p_s} \left[\int_0^p q \varphi_i dp \right] \left[\frac{\partial T}{\partial p} \frac{\partial \tau_j}{\partial p} \right] \left[\frac{\partial B_j}{\partial T} / \frac{\partial B_j}{\partial T_{bj}} \right] dp \quad i=1, \dots, Q$$

$$\psi_{ij} = \int_0^{p_s} \varphi_i \frac{\partial \tau_j}{\partial p} \left[\frac{\partial B_j}{\partial T} / \frac{\partial B_j}{\partial T_{bj}} \right] dp \quad i=Q+1, \dots, L$$

or in matrix form

$$\mathbf{t}_b = \boldsymbol{\psi} \boldsymbol{\alpha}$$

(Mx1) (M x L+1) (L+1 x 1)

A least squares solution suggests that

$$\boldsymbol{\alpha} = (\boldsymbol{\psi}^t \boldsymbol{\psi})^{-1} \boldsymbol{\psi}^t \mathbf{t}_b \approx (\boldsymbol{\psi}^t \boldsymbol{\psi} + \gamma \mathbf{I})^{-1} \boldsymbol{\psi}^t \mathbf{t}_b$$

where $\gamma = .1$ has been incorporated to stabilize the matrix inverse.

There are many reasonable choices for the pressure basis functions $\varphi(p)$. For example empirical orthogonal functions (eigenvectors of the water vapour and temperature profile covariance matrices) can be used in order to include statistical information in the solution. Also the profile weighting functions of the radiative transfer equation can be used. Or gaussian functions that peak in different layers of the atmosphere can be used.

Ancillary information, such as surface observations, are readily incorporated into the profile solutions as additional equations (M+2 equations to solve L unknowns).

$$q_o - q(p_s) = g \sum_{i=1}^Q \alpha_i q(p_s) \varphi_i(p_s)$$

$$T_o - T(p_s) = - \sum_{i=Q+1}^L \alpha_i \varphi_i(p_s)$$

In summary we have the following characteristics (a) the RTE is in perturbation form, (b) δT and δu are expressed as linear expansions of basis functions (EOF or $W(p)$), (c) ancillary observations are used as extra equations, (d) a least squares solution is sought, and (e) a simultaneous temperature and moisture profile solution produces improved moisture determinations. The simultaneous solution addresses the interdependence of water vapour radiance upon temperature and carbon dioxide channel radiance upon water vapour concentration. The dependence of the radiance observations on the surface emissions is accounted for by the inclusion of surface temperature as an unknown. A single matrix solution is computationally efficient compared to an iterative calculation.

5.9 Water Vapour Profile Solutions

The direct physical solution of the RTE provides a simultaneous solution of both the temperature and moisture profiles. It is currently the preferred solution. On the other hand, iterative numerical techniques involve several determinations of each profile separately before self consistent convergence is achieved. The iterative numerical solution for the moisture profile is presented here. It should be viewed as a companion to the iterative numerical solution of the temperature profile presented in section 5.7.

The linear form of the RTE can be written in terms of the precipitable water vapour profile as

$$u_s$$

$$(\Delta T_b)_\lambda = \int_0 (\Delta T) V_\lambda du$$

where

$$V_\lambda = \left[\frac{\partial B_\lambda(T)}{\partial T} \Big|_{T=T_{av}} / \frac{\partial B_\lambda(T)}{\partial T} \Big|_{T=T_{b\lambda}} \right] \frac{\partial \tau_\lambda}{\partial u}$$

and $T_{av}(p)$ represents a mean or initial profile condition.

One manner of solving for the water vapour profile from a set of spectrally independent water vapour radiance observations is to employ one of the linear direct temperature profile solutions discussed earlier. In this case, however, one solves for the function $T(u)$ rather than $T(p)$. Given $T(p)$ from a prior solution of carbon dioxide and/or oxygen channel radiance observations, $u(p)$ can be found by relating $T(p)$ to $T(u)$. The mixing ratio profile, $q(p)$, can then be obtained by taking the vertical derivative of $u(p)$, $q(p) = g \partial u / \partial p$ where g is gravity.

Rosenkranz *et al* (1982) have applied this technique to microwave measurements of water vapour emission. They used the regression solution for both the temperature versus pressure and temperature versus water vapour concentration profiles. The regression solutions have the form

$$T(p_j) = t_o(p_j) + \sum_{i=1}^N t_i(p_i) T_{bi}$$

and

$$T(u_k) = t_o(u_k) + \sum_{m=1}^M t_m(u_k) T_{bm}$$

where T_{bi} are the N brightness temperature observations of oxygen emission and T_{bm} are the M brightness temperature observations of water vapour emission and $t_i(p_j)$ and $t_m(u_k)$ are the regression coefficients corresponding to each pressure and water vapour concentration level. $u(p)$ is found from the intersections of the $T(p)$ and $T(u)$ profiles obtained by interpolation of the discrete values given by the regression solutions. An advantage of the linear regression retrievals is that they minimize the computer requirements for real time data processing since the regression coefficient matrices are predetermined.

Various non-linear iterative retrieval methods for inferring water vapour profiles have been developed and applied to satellite water vapour spectral radiance observations. The formulation shown below follows that given by Smith (1970). Integrating the linear RTE by parts one has

$$T_{b\lambda} - T_{b\lambda}^{(n)} = \int_0^{p_s} [\tau_\lambda(p) - \tau_\lambda^{(n)}(p)] X_\lambda(p) \frac{dp}{p}$$

where

$$X_\lambda(p) = \left[\frac{\partial B_\lambda(T)}{\partial T} \Big|_{T=T_{av}} / \frac{\partial B_\lambda(T)}{\partial T} \Big|_{T=T_{b\lambda}} \right] \frac{\partial T(p)}{\partial \ln p}$$

and the (n) superscript denotes the n^{th} estimate of the true profile. Expanding $\tau_\lambda(p)$ as a logarithmic function of the precipitable water vapour concentration $u(p)$ yields

$$\tau_\lambda(p) - \tau_\lambda^{(n)}(p) = \frac{\partial \tau_\lambda(p)}{\partial \ln u^{(n)}(p)} \ln \frac{u(p)}{u^{(n)}(p)}$$

Using the approximation

$$\frac{\partial \tau_{\lambda}(p)}{\partial \ln u^{(n)}(p)} = \tau_{\lambda}^{(n)}(p) \ln \tau_{\lambda}^{(n)}(p)$$

which is valid for the exponential transmission function, then

$$T_{b\lambda} - T_{b\lambda}^{(n)} = \int_0^{p_s} \ln \frac{u(p)}{u^{(n)}(p)} Y_{\lambda}^{(n)}(p) dp$$

with

$$Y_{\lambda}^{(n)}(p) = \tau_{\lambda}^{(n)}(p) \ln \tau_{\lambda}^{(n)}(p) X_{\lambda}(p)$$

Following the same strategy employed in Smith's generalized iterative temperature profile solution, we realize that from each water vapour channel brightness temperature an estimate of the ratio of the true precipitable water vapour profile with respect to the nth estimate can be calculated by

$$\left[\frac{u(p)}{u^{(n)}(p)} \right]_{\lambda} = \exp \left[\frac{T_{b\lambda} - T_{b\lambda}^{(n)}}{\int_0^{p_s} Y_{\lambda}^{(n)}(p) dp} \right]$$

As in the temperature profile solution, the best average estimate of the precipitable water vapour profile is based upon the weighted mean of all water vapour channel estimates using the weighting function $Y_{\lambda}^{(n)}(p)$.

It follows that the mixing ratio profile $q^{(n+1)}(p)$ can be estimated from $u^{(n+1)}(p_j) / u^{(n)}(p_j)$ and from $q^{(n)}(p_j)$ by using

$$q^{(n+1)}(p) = q^{(n)}(p) \frac{[u^{(n+1)}(p)]}{[u^{(n)}(p)]} + g u^{(n)}(p) \frac{\partial}{\partial p} \frac{[u^{(n+1)}(p)]}{[u^{(n)}(p)]}$$

The advantage of using this expression to compute $q(p)$ is that the second term on the right hand side is small compared to the first term so that numerical errors produced by the vertical differentiation are small.

It should be noted that relative humidity is an immediate by-product of the above derivation. Assuming that the relative humidity is constant within the radiating layer, one can write $\ln(RH/RH^{(n)}) = \ln(u/u^{(n)})$ and thus determine true RH from the nth estimate $RH^{(n)}$.

5.10 Microwave Form of RTE

In the microwave region, the emissivity of the earth atmosphere system is normally less than unity. Thus, there is a reflection contribution from the surface. The radiance emitted from the surface would therefore be given by

$$I_{\lambda}^{sfc} = \varepsilon_{\lambda} B_{\lambda}(T_s) \tau_{\lambda}(p_s) + (1-\varepsilon_{\lambda}) \tau_{\lambda}(p_s) \int_0^{p_s} B_{\lambda}(T(p)) \frac{\partial \tau'_{\lambda}(p)}{\partial \ln p} d \ln p$$

The first term in the right-hand side denotes the surface emission contribution, whereas the second term represents the emission contribution from the entire atmosphere to the surface, which is reflected back to the atmosphere at the same frequency. The transmittance $\tau'_\lambda(p)$ is now expressed with respect to the surface instead of the top of the atmosphere (as $\tau_\lambda(p)$ is). Thus, the upwelling radiance is now expressed as

$$I_\lambda = \varepsilon_\lambda B_\lambda(T_s) \tau_\lambda(p_s) + (1-\varepsilon_\lambda) \tau_\lambda(p_s) \int_0^{p_s} B_\lambda(T(p)) \frac{\partial \tau'_\lambda(p)}{\partial \ln p} d \ln p$$

$$+ \int_{p_s}^0 B_\lambda(T(p)) \frac{\partial \tau_\lambda(p)}{\partial \ln p} d \ln p$$

In the wavelength domain, the Planck function is given by

$$B_\lambda(T) = c_1 / [\lambda^5 (e^{c_2/\lambda T} - 1)] .$$

In the microwave region $c_2/\lambda T \ll 1$, so the Planck function may be approximated by

$$B_\lambda(T) \approx \frac{c_1}{c_2} \frac{T}{\lambda^4} ;$$

the Planck radiance is linearly proportional to the temperature. Analogous to the above approximation, we may define an equivalent brightness temperature T_b such that

$$I_\lambda = \frac{c_1}{c_2} \frac{T_b}{\lambda^4} .$$

Thus, the microwave radiative transfer equation may now be written in terms of temperature

$$T_{b\lambda} = \varepsilon_\lambda T_s \tau_\lambda(p_s) + (1-\varepsilon_\lambda) \tau_\lambda(p_s) \int_0^{p_s} T(p) \frac{\partial \tau'_\lambda(p)}{\partial \ln p} d \ln p$$

$$+ \int_{p_s}^0 T(p) \frac{\partial \tau_\lambda(p)}{\partial \ln p} d \ln p .$$

The transmittance to the surface can be expressed in terms of transmittance to the top of the atmosphere by remembering

$$\tau'_\lambda(p) = \exp \left[- \frac{1}{g} \int_0^{p_s} k_\lambda(p) g(p) dp \right]$$

$$= \exp \left[- \int_0^{p_s} \frac{p}{g} + \int_0^p \right]$$

$$= \tau_\lambda(p_s) / \tau_\lambda(p) .$$

So

$$\frac{\partial \tau'_\lambda(p)}{\partial \ln p} = - \frac{\tau_\lambda(p_s)}{(\tau_\lambda(p))^2} \frac{\partial \tau_\lambda(p)}{\partial \ln p}.$$

And thus to achieve a form similar to that of the infrared RTE, we write

$$T_{b\lambda} = \varepsilon_\lambda T_s(p_s) \tau_\lambda(p_s) + \int_{p_s}^0 T(p) F_\lambda(p) \frac{\partial \tau_\lambda(p)}{\partial \ln p} d \ln p$$

where

$$F_\lambda(p) = \left\{ 1 + (1 - \varepsilon_\lambda) \left[\frac{\tau_\lambda(p_s)}{\tau_\lambda(p)} \right]^2 \right\}.$$

A special problem area in the use of microwave for atmospheric sounding from a satellite platform is surface emissivity. In the microwave spectrum, emissivity values of the earth's surface vary over a considerable range, from about 0.4 to 1.0. The emissivity of the sea surface typically ranges between 0.4 and 0.5, depending upon such variables as salinity, sea ice, surface roughness, and sea foam. In addition, there is a frequency dependence with higher frequencies displaying higher emissivity values. Over land, the emissivity depends on the moisture content of the soil. Wetting of a soil surface results in a rapid decrease in emissivity. The emissivity of dry soil is on the order of 0.95 to 0.97, while for wet bare soil it is about 0.80 to 0.90, depending on the frequency. The surface emissivity appearing in the first term has a significant effect on the brightness temperature value.

The basic concept of inferring atmospheric temperatures from satellite observations of thermal microwave emission in the oxygen spectrum was developed by Meeks and Lilley (1963) in whose work the microwave weighting functions were first calculated. The prime advantage of microwave over infrared temperature sounders is that the longer wavelength microwaves are much less influenced by clouds and precipitation. Consequently, microwave sounders can be effectively utilized to infer atmospheric temperatures in all weather conditions. We will not pursue microwave retrievals in this course, except to say that the techniques are similar to those for infrared retrieval.

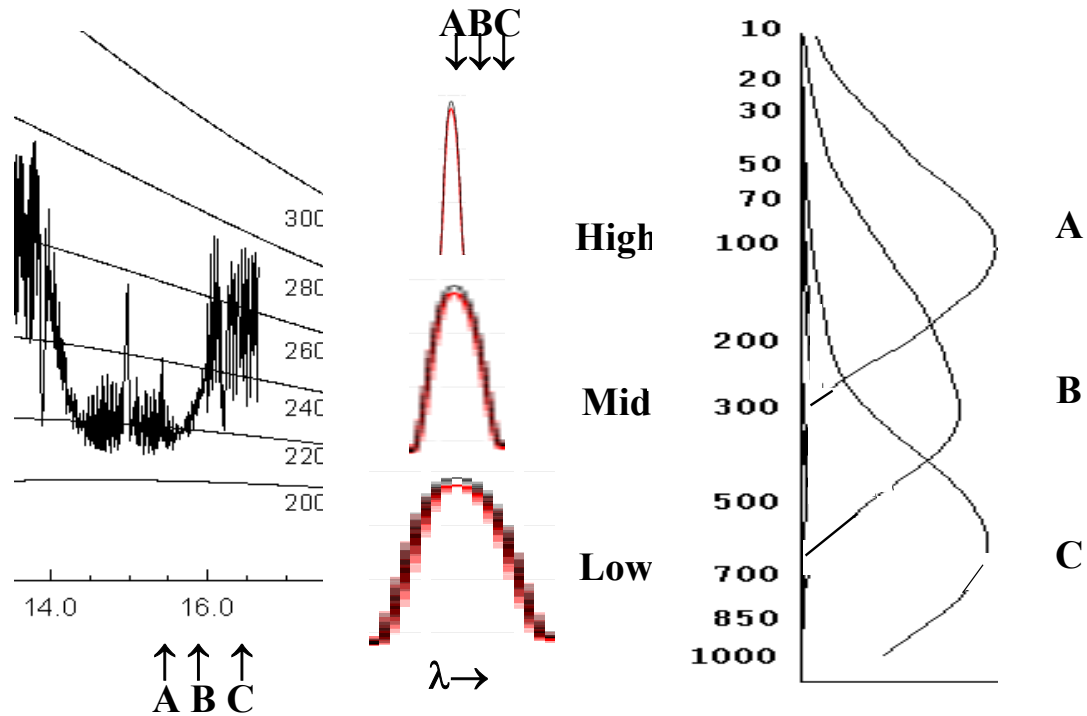


Figure 5.1: Location of three spectral bands in the CO₂ absorption band at 15 μm (left), pressure broadening of the absorption line from high to low in the atmosphere with the spectral band locations indicated (center), and vertical distribution of the weighting functions for the three spectral bands (right).

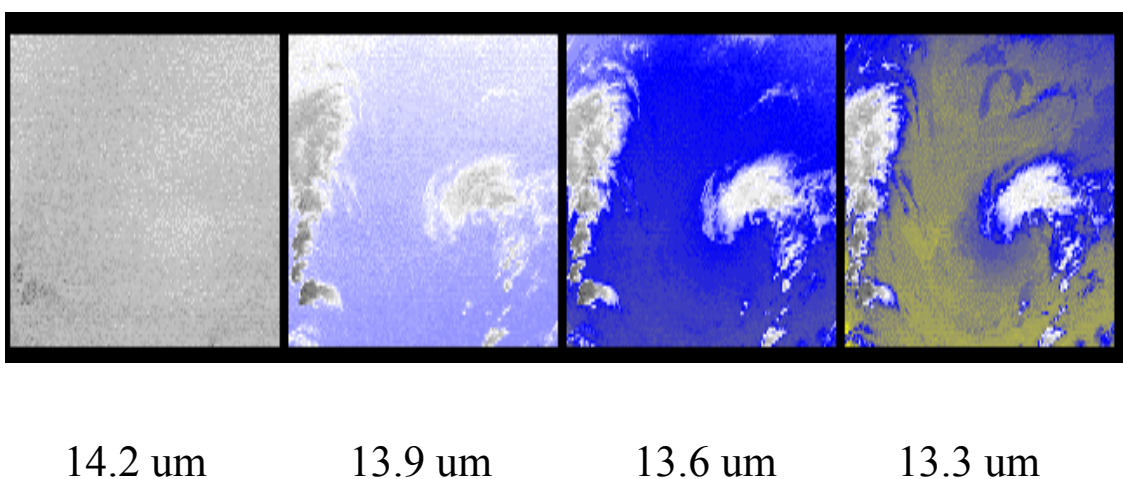


Figure 5.2: Images of measurements in the CO₂ spectral region from very opaque (left) to nearly transparent (right).

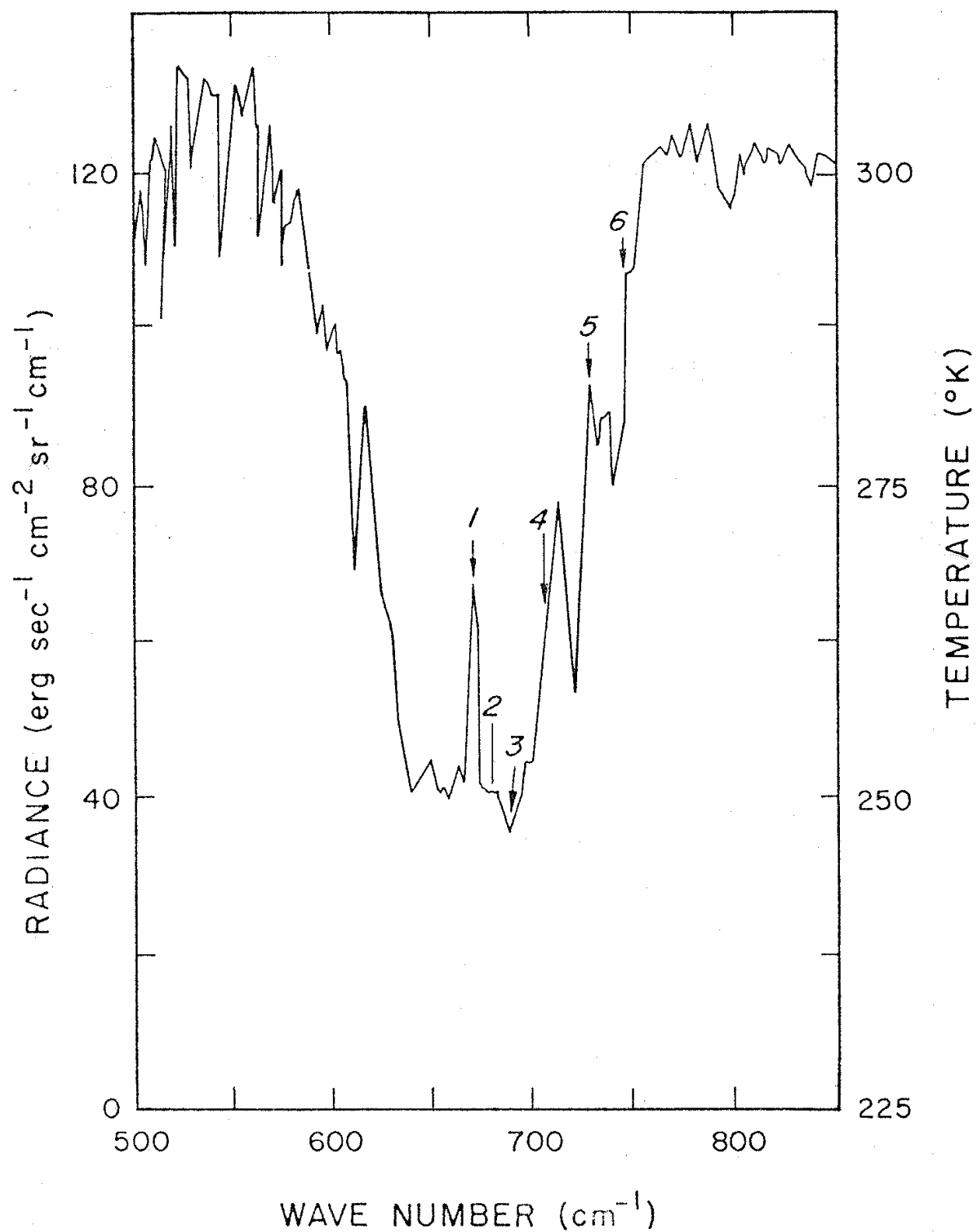


Figure 5.3: Outgoing radiance in terms of black body temperature in the vicinity of 15 μ m CO₂ band observed by the IRIS on Nimbus IV. The arrows denote the spectral regions sampled by the VTPR instrument.

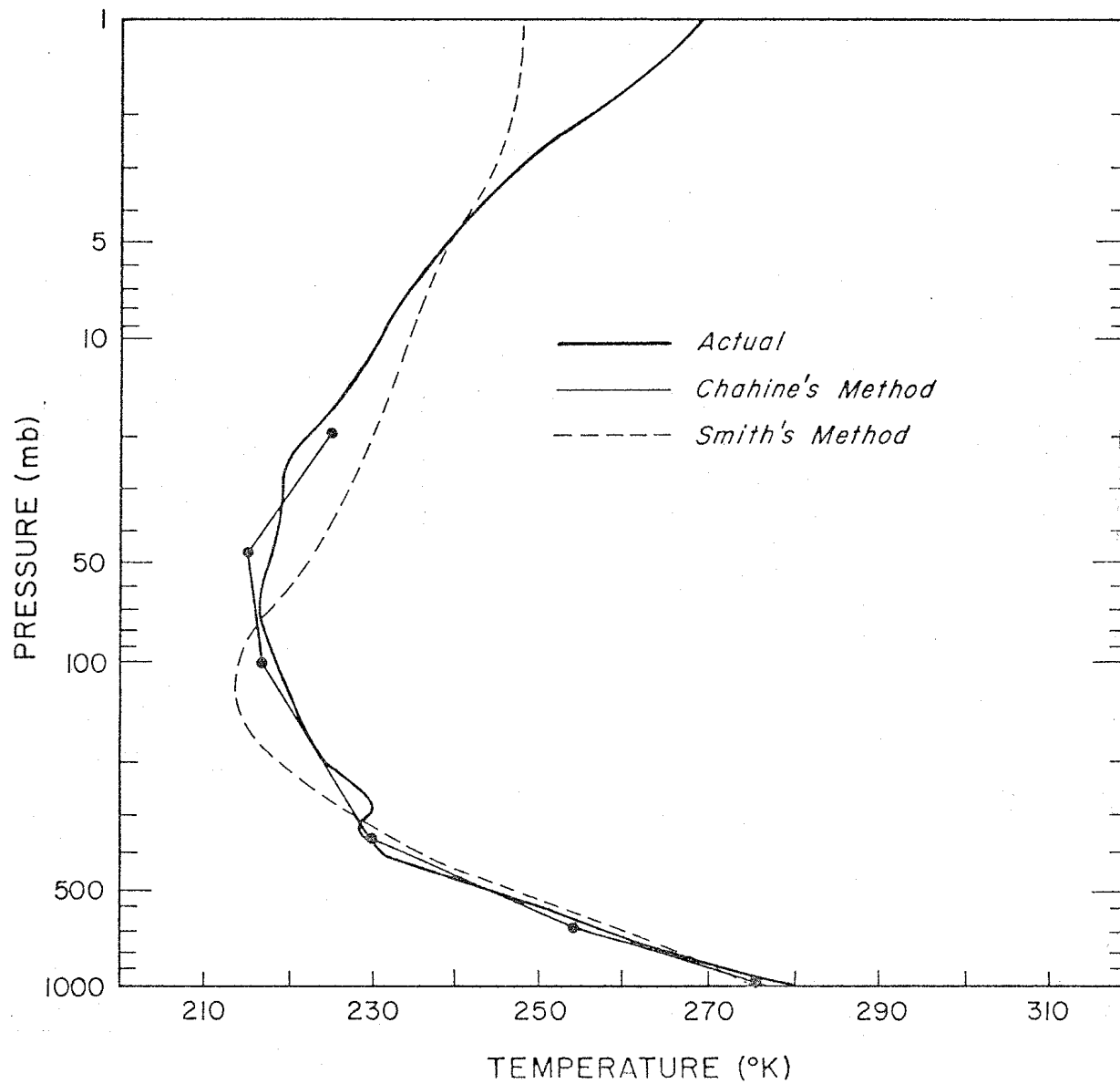


Figure 5.4: Temperature retrieval using Chahine's relaxation and Smith's iterative methods for the VTPR channels.